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On p-Quotients for Spin Characters

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INTRODUCTION

It is a well-known combinatorial fact (see, e.g., [3]) that given positive integers p, n a partition λ of n is uniquely determined by its p-core $\lambda_{(p)}$ and its p-quotient $\lambda^{(p)} = (\lambda_0, \lambda_1, ..., \lambda_{p-1})$. Here $\lambda_{(p)}$ is a partition without p-hooks and $\lambda_0, \lambda_1, ..., \lambda_{p-1}$ are partitions such that

$$n = |\lambda| = |\lambda_{(p)}| + pw,$$

where

$$w = |\lambda_0| + |\lambda_1| + \cdots + |\lambda_{p-1}|.$$

This fact has important implications for the actual computation of the values of the irreducible character $[\lambda]$ of S_n on p-singular elements.

We call $[\lambda_{(p)}]$ the *p-core character* and $[\lambda^{(p)}] = ([\lambda_0] \otimes \cdots \otimes [\lambda_{p-1}])^{S_w}$ the *p-quotient character* related to λ. If $\rho \vdash w$, $\rho' \vdash n - pw$ then

(i) $[\lambda](p\rho \cdot \rho') = \delta_p(\lambda)[\lambda^{(p)}](\rho)[\lambda_{(p)}](\rho')$, where $\delta_p(\lambda)$ is a sign (see [1, 12]).

The main purpose of this paper is to prove a similar result for the spin characters of a covering group $\hat{S}_n$ of S_n.

The spin characters of $\hat{S}_n$ are indexed by 2-regular partitions of n, that is, partitions with distinct parts. As the first author [8] realized a long time

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ago, the so-called bars for 2-regular partitions play a similar role for spin characters as do the hooks for ordinary characters of S_n. In Section 1 we describe briefly the theory of bars including the p-bar core and and p-bar quotient of a 2-regular partition and prove a Murnaghan–Nakayama-type formula for spin characters. In Section 2 we describe in a slightly different way the relative (p-) sign of a partition (as defined by Farahat [1]) and introduce a similar concept for 2-regular partitions, which is much more complicated. We also consider branching coefficients which are relevant for the computation of certain powers of 2 occurring in our formulae. As a final preparation for our main result we consider in Section 3 a combinatorial result on leg lengths (which is due to Robinson [12] in the case of partitions). Then in Section 4 we define the p-bar quotient characters and prove a result analogous to (i). New features in our formula are the powers of 2 and the different way of calculating the relative sign mentioned above.

1. Preliminaries and the Murnaghan–Nakayama Formula for Spin Characters

Let λ be a *bar partition* of n, that is, a 2-regular partition

$$\lambda = (a_1, a_2, ..., a_m) \qquad \text{with} \quad a_1 > a_2 > \cdots > a_m > 0.$$

We write $\lambda \succ\!\!- n$ and $|\lambda| = n$ and we denote the number of parts in λ by $m(\lambda)$. Let $S(\lambda)$ be the *shifted Young diagram* of λ, that is, $S(\lambda)$ is obtained from the Young diagram of λ by moving the ith row of λ $i-1$ positions to the right. Alternatively, $S(\lambda)$ is the part above the diagonal in the Young diagram corresponding to the partition

$$\tilde{\lambda} = (a_1, ..., a_m; a_1 - 1, ..., a_m - 1)$$

of $2n$ expressed in the Frobenius notation (see, e.g., [4]). The jth node in the ith row of $S(\lambda)$ is called the (i, j)-node. The *bar length* $\bar{h}_{ij}^{\lambda}$ of the (i, j)-node of $S(\lambda)$ is defined to be the hook length of that node in the diagram of $\tilde{\lambda}$. Let $\bar{l}_{ij}^{\lambda}$ denote the leg length of the corresponding hook in $\tilde{\lambda}$.

The bar lengths $\bar{h}_{ij}^{\lambda}$, $j = 1, 2, ..., a_i$, in the ith row of $S(\lambda)$ are the elements of the following set:

$$\{a_i + a_k \mid k = i+1, ..., m\} \cup \{\{a_i, a_i - 1, ..., 2, 1\} \setminus \{a_i - a_k \mid k = i+1, ..., m\}\}.$$

To each node in $S(\lambda)$ we associate a *bar* as follows:

(i) if $i + j \geqslant m + 2$, then the (i, j)-bar $\bar{H}_{ij}^{\lambda}$ consists of the last $\bar{h}_{ij}^{\lambda}$ in the ith row of $S(\lambda)$ and is called a bar of type 1;

(ii) if $i+j=m+1$, then the (i, j)-bar $\bar{H}_{ij}^{\lambda}$ consists of the ith row of $S(\lambda)$ and is called a bar of type 2;

(iii) if $i+j \leqslant m$, then the (i, j)-bar $\bar{H}_{ij}^{\lambda}$ consists of all the nodes in the ith and $(i+j)$th rows of $S(\lambda)$, and is called a bar of type 3.

Type 1 and 2 (i, j)-bars are called *unmixed* bars and type 3 (i, j)-bars are called *mixed* bars.

For the leg lengths we have

$$\bar{l}_{ij}^{\lambda} = \begin{cases} |\{k \mid a_i > a_k > a_i - \bar{h}_{ij}^{\lambda}\}| & \text{for bars of type 1 and 2,} \\ a_{i+j} + |\{k \mid a_i > a_k > a_{i+j}\}|. \end{cases}$$

If the nodes of a bar are removed from $S(\lambda)$, and the rows of the resulting diagram are rearranged so that they correspond to a bar partition, then a shifted diagram $S(\mu)$ is obtained. If the (i, j)-bar is removed, then we write $\mu = \lambda \backslash \bar{H}_{ij}^{\lambda}$ and $\mu \succ\!\!- n - \bar{h}_{ij}^{\lambda}$. Humphreys [2] and Morris and Yaseen [9] have shown that the removal of a bar from $S(\lambda)$ is equivalent to the removal of a pair of hooks of the same length from $\tilde{\lambda}$.

Let p be an odd number. A bar partition λ may be represented on a *p-abacus* as follows [9, 10]. The abacus has p runners numbered $0, 1, ..., p-1$ from left to right. The rows of the abacus are numbered $0, 1, 2, ...$. The bead configuration of the abacus corresponding to $\lambda = (a_1, ..., a_m)$ then has a bead in the jth row of the ith runner if and only if $a_k = jp + i$ for some k. For $1 \leqslant i \leqslant t = (p-1)/2$, the runners i and $(p-i)$ are called *conjugate runners*. A bar of length p is called a *p-bar*.

The removal of a p-bar is registered on the abacus as follows.

Type 1: Slide a bead one position up on the same runner.
Type 2: Remove the bead in the position $(1, 0)$.
Type 3: Remove the two beads in the positions $(0, i)$ and $(0, p-i)$ for some i $(1 \leqslant i \leqslant t)$, i.e., remove the beads in the zeroth row on conjugate runners.

The *$\bar{p}$-core* $\lambda_{(\bar{p})}$ of λ is the partition obtained by removing recursively all possible p-bars from λ. Then $\lambda_{(\bar{p})}$ is uniquely determined (see [9] or [10] for a proof).

The *p-bar quotient* or *$\bar{p}$-quotient* $\lambda^{(\bar{p})}$ has been defined in two alternative but equivalent ways by Morris and Yaseen [9] and Olsson [10]. We give the definition of [9] as this is more easily presented directly. Then

$$\lambda^{(\bar{p})} = (\lambda_0, \lambda_1, ..., \lambda_t)$$

is a $(t+1)$-tuple of partitions defined in terms of the p-abacus representation of λ as follows:

(i) if there are beads on runner 0 in the rows $\alpha_1, \alpha_2, ..., \alpha_r$, where $\alpha_1 > \alpha_2 > \cdots > \alpha_r > 0$, then

$$\lambda_0 = (\alpha_1, \alpha_2, ..., \alpha_r);$$

(ii) for $1 \leqslant i \leqslant t$, if there are beads on the conjugate runners i and $(p-i)$ in the rows $\alpha_1, \alpha_2, ..., \alpha_r$, $\alpha_1 > \alpha_2 > \cdots > \alpha_r \geqslant 0$, and $\beta_1, \beta_2, ..., \beta_s$, $\beta_1 > \beta_2 > \cdots > \beta_s \geqslant 0$, respectively, and assuming without loss of generality that $r \geqslant s$, and if $u = r - s$, then

$$\lambda_i = ((\alpha_1 - u + 1, \alpha_2 - u + 2, ..., \alpha_u); (\alpha_{u+1}, ..., \alpha_r; \beta_1, ..., \beta_s)),$$

where in λ_i the first part is in the ordinary notation for partitions and the second part is the Frobenius notation. The $\bar{p}$-weight of λ is defined to be $w_{\bar{p}}(\lambda) = \sum_{i=0}^{t} |\lambda_i|$. Note that

$$|\lambda| = |\lambda_{(\bar{p})}| + p w_{\bar{p}}(\lambda).$$

For example, if $\lambda = (13, 11, 10, 9, 7, 6, 4, 2)$ and $p = 3$, the bead configuration is

	0	1	2
0	0	1	②
1	3	④	5
2	⑥	⑦	8
3	⑨	⑩	⑪
4	12	⑬	14

and thus $\lambda_0 = (32)$ and $\lambda_1 = ((3^2); (21; 30)) = (3^4 1^2)$ and hence

$$\lambda^{(3)} = (32; 3^4 1^2)$$

and $\lambda_{(3)} = (41)$.

The main point to note is:

THEOREM (1.1). *A bar partition determines and is uniquely determined by its $\bar{p}$-core and its $\bar{p}$-quotient.*

Schur [13] showed that the symmetric group S_n has a representation group $\hat{S}_n$ of order $2 \cdot n!$, that is, $\hat{S}_n$ has a central subgroup $\mathbb{Z}_2 = \{1, -1\}$ such that $\hat{S}_n/\mathbb{Z}_2 \cong S_n$. The group $\hat{S}_n$ has two types of representations.

(1) The representations of $\hat{S}_n$ which are equal on the central elements 1 and -1. These are the ordinary representations of S_n which are indexed by partitions of n. If λ is a partition of n ($\lambda \vdash n$) let $[\lambda]$ denote the ordinary character of S_n (or $\hat{S}_n$) corresponding to λ.

(2) The representations of $\hat{S}_n$ which represent the element -1 by the negative of the identity matrix. These representations are indexed by bar partitions of n and are called spin representations and their characters spin characters. Let $\lambda = (a_1, ..., a_m) \succ\!\!- n$. If $n-m$ is odd then there are two irreducible spin characters denoted by $\langle\lambda\rangle$ and $\langle\lambda\rangle'$ which are associate characters. If $n-m$ is even there is one irreducible spin character denoted by $\langle\lambda\rangle$ which is a self-associate character. However, we shall consider only character values on elements of odd order in S_n, which are indexed canonically by partitions of n with all parts odd. As $\langle\lambda\rangle$ and $\langle\lambda\rangle'$ are equal on these elements, from now on we shall consider $\langle\lambda\rangle$ only. Let $P^0(n)$ denote the set of partitions of n with all parts odd. If $\pi \in P^0(k)$ and $\mu \in P^0(l)$, let $\pi \circ \mu \in P^0(k+l)$ be the partition having the parts of π and μ as parts (e.g., if $\pi = (3, 1)$, $\mu = (5, 3)$, then $\pi \circ \mu = (5, 3^2, 1)$). If χ is a character of $\hat{S}_n$ (ordinary or spin), then $\chi(\rho)$ is the character value of χ on an element of odd order of type ρ.

There are no explicit formulae in general for the $\langle\lambda\rangle(\pi)$, $\lambda \succ\!\!- n$, $\pi \in P^0(n)$. However, for the special case $(n) \succ\!\!- n$, Schur [13] or Morris [5] have proved the following.

LEMMA (1.2). *If $\pi \in P^0(n)$, then*

$$\langle n\rangle(\pi) = 2^{[(m(\pi)-1)/2]},$$

where $m(\pi)$ is the number of parts in π.

Here [ ] stands for the "integral part of."

For the general case, Schur [13] introduced some symmetric functions Q_λ ($\lambda \succ\!\!- n$) which can be used to describe the spin characters $\langle\lambda\rangle$. These are the special cases corresponding to $t = -1$ of the Hall–Littlewood polynomials $Q_\lambda(\mathbf{x}, t)$ (for a complete description see Macdonald [4]), where $\mathbf{x} = (x_1, x_2, ...)$ is a countable infinite set of variables and t is an indeterminate. (In fact, Schur functions correspond to the case $t = 0$ and are used to "generate" the ordinary characters $[\lambda]$.) If $\lambda \succ\!\!- n$, we define $Q_\lambda := Q_\lambda(\mathbf{x}, -1)$. Let s_r ($r \geqslant 1$) be the rth power sum of the x_i, that is,

$$s_r = \sum_{i=1}^{\infty} x_i^r. \tag{1}$$

If $\pi=(1^{\pi_1}3^{\pi_3}\cdots)\in P^0(n)$, let $s_\pi=s_1^{\pi_1}s_3^{\pi_3}, ..., z_\pi=1^{\pi_1}3^{\pi_3}, ...,$ and if $\lambda \succ\!\!- n$ define

$$\varepsilon(\lambda):=\begin{cases}0 & \text{if } \ |\lambda|-m(\lambda) \text{ is even,}\\ 1 & \text{if } \ |\lambda|-m(\lambda) \text{ is odd,}\end{cases} \tag{2}$$

and we say that λ has *even* and *odd* parity, respectively.

Then if $\lambda \succ\!\!- n$, Schur [13] proved that

$$Q_\lambda=\sum_{\pi\in P^0(n)} 2^{(m(\lambda)+m(\pi)+\varepsilon(\lambda))/2} z_\pi^{-1}\langle\lambda\rangle(\pi)s_\pi. \tag{3}$$

(Note that in contrast to Macdonald [4] we have used s_r rather than p_r for power series for a reason which will become apparent later.)

As is shown in Macdonald [4, p. 107] and Morris [7], the definition of the Q_λ may be extended to any sequence $\mu=(b_1, b_2, ..., b_l)$ where the b_i are integers, positive, negative, or zero, which are not necessarily in descending order. Such a Q_μ can be shown to be a multiple of a Q_λ, where $\lambda \succ\!\!- n$ as follows:

In the general case for t arbitrary, Morris [7] or Macdonald [4] have shown that

$$\text{if } \ \lambda_l<0, \text{ then } Q_\lambda=0$$

and

$$\text{if } \ s<r, \text{ then}$$

$$Q_{(s,r)}=\begin{cases} tQ_{(r,s)}+\sum_{i=1}^{m}(t^{i+1}-t^{i-1})\,Q_{(r-i,s+i)} & \text{if } \ r-s=2m+1,\\ tQ_{(r,s)}+\sum_{i=1}^{m-1}(t^{i+1}-t^{i-1})\,Q_{(r-i,s+i)} & \\ \qquad +(t^m-t^{m-1})\,Q_{(r-m,s+m)} & \text{if } \ r-s=2m.\end{cases}$$

Thus, if $t=-1$, and $s<r$, we have

$$Q_{(s,r)}=\begin{cases}-Q_{(r,s)} & \text{if } \ r-s=2m+1,\\ -Q_{(r,s)}+2(-1)^m\,Q_{(s+m,s+m)} & \text{if } \ r-s=2m.\end{cases}$$

This implies that if $s<r$

$$Q_{(s,r)}=-Q_{(r,s)} \qquad \text{if } \ s\neq -r \tag{4}$$

and

$$Q_{(-r,r)}=2(-1)^r. \tag{5}$$

In particular, we have $Q_{(r,r)} = 0$. These formulae may be extended for any two consecutive terms of a sequence μ and may be used repeatedly to express any Q_μ as a multiple of a Q_λ with $\lambda \succ\!\!- n$.

Later we need to apply these rules to a sequence μ obtained from a bar partition $\lambda = (a_1, ..., a_m)$ by subtracting an odd integer p from any one of the parts a_i. Let $\mu = (a_1, ..., a_i - p, ..., a_m)$ for some i.

If $a_i - p = a_k$ for some k then $Q_\mu = 0$. In the following, $\hat{a}_i$ means that the term a_i is deleted for some i. There are three cases to be considered.

Case 1. Let k be the least integer such that

$$a_k > a_i - p > a_{k+1}.$$

Then, by repeated application of (4), we have

$$Q_\mu = (-1)^{k-i} Q_\nu, \tag{6}$$

where $\nu = (a_1, ..., \hat{a}_i, ..., a_k, a_i - p, a_{k+1}, ..., a_m)$ is the bar partition of $n - p$ obtained by removing a p-bar of type 1 from the ith row of λ and where by (1), $k - i$ is clearly the leg length of the corresponding p-hook in $\tilde{\lambda}$.

Case 2. If $a_i - p = 0$ for some i, then again by repeated application of (4), we have

$$Q_\mu = (-1)^{m-i} Q_\nu, \tag{7}$$

where $\nu = (a_1, ..., \hat{a}_i, ..., a_m)$ is the bar partition of $n - p$ obtained by removing a p-bar of type 2 comprising the ith row of λ from λ and where by (1), $m - i$ is the leg length of the corresponding p-hook in $\tilde{\lambda}$.

Case 3. If $a_i - p = -a_k$ for some $k > i$ (that is, $a_i + a_k = p$), then by repeated application of (4) and (5), we have

$$Q_\mu = 2(-1)^{a_k + k - i - 1} Q_\nu, \tag{8}$$

where $\nu = (a_1, ..., \hat{a}_i, ..., \hat{a}_k, ..., a_m)$ is the bar partition of $n - p$ obtained by removing a p-bar of type 3 comprising the ith and kth rows of λ from λ and where by (1), $a_k + k - i - 1$ is the leg length of the corresponding p-hook in $\tilde{\lambda}$.

These three cases may therefore be summarised as follows:

Let $\mu = (a_1, ..., a_i - p, ..., a_m)$ and let $\lambda \backslash \bar{H}$ be the corresponding bar partition of $n - p$ obtained by removing a p-bar $\bar{H}$ from λ (or a p-hook from $\tilde{\lambda}$). Let $l(\bar{H})$ be the leg length of this corresponding p-hook in $\tilde{\lambda}$ and let

$$k(\bar{H}) = \begin{cases} 0 & \text{if } \bar{H} \text{ is unmixed,} \\ 1 & \text{if } \bar{H} \text{ is mixed.} \end{cases}$$

Then

$$Q_\mu = 2^{k(\bar{H})}(-1)^{l(\bar{H})} Q_{\lambda \backslash \bar{H}}. \tag{9}$$

This is now used to obtain a Murnaghan–Nakayama-type formula for spin characters of $\hat{S}_n$. An earlier version was proved by Morris [6, 8]. We present a complete proof of a refined version which is more suitable for application in Section 4 of this paper.

Put $q_n = Q_n$. Then from Lemma 1.2 and (3), it follows that

$$q_n = \sum_{\pi \in P^0(n)} 2^{m(\pi)} z_\pi^{-1} s_\pi. \tag{10}$$

If $\lambda = (a_1, ..., a_m) \vdash n$, put $q_\lambda = q_{a_1} \cdots q_{a_m}$. Then q_n $(n \geqslant 0)$, q_λ $(\lambda \vdash n)$, and Q_μ $(\mu \succ\!\!- n)$ may be regarded as polynomials in the ring $\mathbb{Q}[s_1, s_2, ...]$. If $p \leqslant n$ is an odd integer, then we may differentiate q_n formally with respect to s_p and it is easily verified that

$$p \frac{\partial}{\partial s_p} q_n = 2q_{n-p}. \tag{11}$$

By applying Macdonald [4, III(2.15)] in the special case $t = -1$, we have

$$Q_\lambda = \prod_{1 \leqslant j < k \leqslant m} \frac{1 - R_{jk}}{1 + R_{jk}} q_\lambda,$$

where the R_{jk} are raising operators

$$R_{jk} q_\lambda = q_{(a_1, ..., a_j + 1, ..., a_k - 1, ..., a_m)}.$$

Thus

$$Q_\lambda = \sum_R \psi_R q_{R\lambda},$$

where $\psi_R \in \mathbb{Z}$ and the sum is over all raising operators R and $q_{R\lambda} = 0$ if any part of $R\lambda$ is negative. Replacing λ by $\lambda - p\varepsilon_i$, where $\varepsilon_i = (0, ..., 0, 1, 0, ..., 0)$ with 1 in the ith place, we obtain

$$\sum_{i=1}^m Q_{\lambda - p\varepsilon_i} = \sum_R \psi_R \sum_{i=1}^m q_{R(\lambda - p\varepsilon_i)}$$

and by (11), we have

$$2\sum_{i=1}^{m} Q_{\lambda-p\varepsilon_i}=\sum_R \psi_R \sum_{i=1}^{m} 2q_{R\lambda-p\varepsilon_i}$$
$$=p\sum_R \psi_R \sum_{i=1}^{m} q_{(R\lambda)_1}\cdots(\partial/\partial s_p)q_{(R\lambda)_i}\cdots q_{(R\lambda)_m}$$
$$=p\frac{\partial}{\partial s_p}Q_\lambda,$$

where we have put $R\lambda=((R\lambda)_1, ..., (R\lambda)_m)$.

Then, using (9), we obtain

$$p\frac{\partial}{\partial s_p}Q_\lambda=2\sum_{\bar{H}} 2^{k(\bar{H})}(-1)^{l(\bar{H})}Q_{\lambda\backslash\bar{H}}, \tag{12}$$

where the summation is over all the p-bars $\bar{H}$ in λ.

By substituting (3) on both sides of (12), if $\pi\in P^0(n-p)$, we obtain

$$\langle\lambda\rangle((p)\circ\pi)=\sum_{\bar{H}} 2^{k(\bar{H})}(-1)^{l(\bar{H})}\,2^{(1/2)(m(\lambda\backslash\bar{H})-m(\lambda)+\varepsilon(\lambda\backslash\bar{H})-\varepsilon(\lambda)+1)}\langle\lambda\backslash\bar{H}\rangle(\pi). \tag{13}$$

The power of 2 which appears in this formula can be simplified. The three types of p-bar are considered separately. For the time being we write $v=\lambda\backslash\bar{H}$ and $k_{\lambda v}$ for the corresponding $k(\bar{H})$.

Type 1. In this case, v is obtained from λ by removing an unmixed p-bar made up of the last $l<a_i$ nodes in the ith row of λ. Thus $k_{\lambda v}=0$, $m(\lambda)-m(v)=0$. If $n-m(\lambda)$ is even, then $(n-p)-m(\lambda)$ is odd and thus $\varepsilon(\lambda)=0$ and $\varepsilon(\mu)=1$, giving 2. However, if $n-m(\lambda)$ is odd, we obtain $2^0=1$.

Type 2. In this case, v is obtained from λ by removing an unmixed p-bar made up of one of the $a_i=p$. Thus $k_{\lambda v}=0$, $m(\lambda)-m(v)=1$, and $n-m(\lambda)$ and $(n-p)-m(\mu)$ are either both even or both odd and so $\varepsilon(\lambda)-\varepsilon(v)=0$. Thus we obtain overall $2^0=1$.

Type 3. In this case, v is obtained from λ by removing a mixed p-bar; thus $k_{\lambda v}=1$, $m(\lambda)-m(v)=2$. If $n-m(\lambda)$ is even, then $(n-p)-m(v)$ is odd and so $\varepsilon(\lambda)=0$ and $\varepsilon(v)=1$, giving an overall 2. However, if $n-m(\lambda)$ is odd, then $(n-p)-m(\mu)$ is even and so $\varepsilon(\lambda)=1$ and $\varepsilon(v)=0$, giving an overall $2^0=1$.

Thus we see that a 2 *appears in* (13) *if and only if* $\varepsilon(v)-\varepsilon(\lambda)=1$. We have therefore proved the following.

THEOREM (1.3) (Murnaghan–Nakayama formula for spin characters). *Let $l(\bar{H})$ be the leg length of the p-bar $\bar{H}$ removed from λ to give $\lambda\backslash\bar{H}$ and let*

$$m(\bar{H})=\begin{cases}1 & \text{if } \ \varepsilon(\lambda\backslash\bar{H})-\varepsilon(\lambda)=1,\\ 0 & \text{otherwise.}\end{cases}$$

If $\pi\in P^0(n-p)$, then

$$\langle\lambda\rangle((p)\circ\pi)=\sum_{\bar{H}}(-1)^{l(\bar{H})}\,2^{m(\bar{H})}\langle\lambda\backslash\bar{H}\rangle(\pi),$$

where the summation is over all the p-bars $\bar{H}$ in λ.

The following corollary is an easy consequence.

COROLLARY (1.4). *If $\pi\in P^0(n-p)$, then*

$$\langle n\rangle((p)\circ\pi)=2^d\langle n-p\rangle(\pi),$$

where

$$d=\begin{cases}0 & \text{if } \ n \text{ is even},\\ 1 & \text{if } \ n \text{ is odd}.\end{cases}$$

For later application, if $n\in\mathbb{N}$, we shall need to consider the character $\langle\hat{n}\rangle$ defined as

$$\langle\hat{n}\rangle=\begin{cases}\langle n\rangle+\langle n\rangle' & \text{if } \ n \text{ is even},\\ \langle n\rangle & \text{if } \ n \text{ is odd}.\end{cases}$$

Then, it follows trivially that we have

COROLLARY (1.5). *If $\pi\in P^0(n-p)$, then*

$$\langle\hat{n}\rangle((p)\circ\pi)=2^{d'}\langle\widehat{n-p}\rangle(\pi),$$

where $d'=1$ if n is even and $d'=0$ if n is odd.

If $\lambda\vdash n$, we further define

$$\langle\hat{\lambda}\rangle:=\langle\hat{n}\rangle[\lambda],$$

then $\langle\hat{\lambda}\rangle$ is a character of $\hat{S}_n$ and by combining the Murnaghan–Nakayama formula for ordinary characters of $\hat{S}_n$ with (1.5) we easily obtain

LEMMA (1.6). *Let $\lambda \vdash n$, $\pi \in P^0(n-p)$, p odd. Then*

$$\langle\hat{\lambda}\rangle((p)\circ\pi) = 2^{d'} \sum_{H} (-1)^{l(H)} \langle\widehat{\lambda\backslash H}\rangle(\pi),$$

where the summation is over the set of p-hooks H in λ, $l(H)$ is the leg length of H, and d' is as above.

2. The Relative Sign and the Branching Coefficient

We describe the "relative sign" for partitions (defined by Farahat [1], see also [3, p. 82] for a special case) in a somewhat different form and then introduce a similar concept for bar partitions. The latter is considerably more complicated.

Let X be a β-set (i.e., a finite subset of $\mathbb{N}_0$). If $X = \{x_1, ..., x_r\}$, $x_1 > x_2 > \cdots > x_r$, we call X a β-set for the partition $\lambda = (x_1 - (r-1), x_2 - (r-2), ..., x_r)$. Then $X^{+1} := \{x_1 + 1, ..., x_r + 1, 0\}$ is a β-set for the same partition. We number the elements of X in two ways with the numbers $1, 2, ..., |X|$.

In the *natural numbering* the elements of X are numbered according to their size in increasing order. (Thus c is numbered lower than d if and only if $c < d$.)

Next, let p be an arbitrary positive integer. We represent the elements of X as beads on the p-abacus (as in [3, pp. 78–79]). (So there is a bead in the ath row of the ith runner $(0 \leqslant i \leqslant p-1)$ if and only if $ap + i \in X$.) The jth *layer*, $j \geqslant 1$, consists of those beads which have $(j-1)$ beads above them on their respective runners. In the *p-numbering* the element of X represented by the jth bead in the ith runner is numbered before the element represented by the j_1th bead on the i_1th runner if and only if

$$j < j_1 \quad \text{or} \quad j = j_1 \quad \text{and} \quad i < i_1$$

(i.e., according to the layers!).

When we compare the natural numbering of X with the p-numbering of X we get a permutation $\pi_p(X)$ of $\{1, 2, ..., |X|\}$, whose *sign* is denoted by $\delta_p(X)$. It is easily verified that $\delta_p(X) = \delta_p(X^{+1})$ and so if X is any β-set for the partition λ we may define

$$\delta_p(\lambda) = \delta_p(X),$$

the *p-sign of* λ.

EXAMPLE. Let $p=3$ and consider the β-set $X=\{9, 8, 7, 4, 2, 1\}$ for the partition $\lambda=(4, 4, 4, 2)$. The 3-abacus of X is

$$\begin{array}{ccc} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \end{array}$$

Then the first layer is 0, 1, 8, the second layer is 9, 4, and the third layer is 7. Thus

$$\text{Natural numbering: } \begin{array}{ccc} 0_1 & 1_2 & 2 \\ 3 & 4_3 & 5 \\ 6 & 7_4 & 8_5 \\ 9_6 & 10 & 11 \end{array} \qquad \text{3-numbering: } \begin{array}{ccc} 0_1 & 1_2 & 2 \\ 3 & 4_5 & 5 \\ 6 & 7_6 & 8_3 \\ 9_4 & 10 & \end{array}$$

and so

$$\pi_3(X)=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 6 & 3 & 4 \end{pmatrix}=(35)(46)$$

and

$$\delta_3(X)=\delta_3(\lambda)=1.$$

Suppose that the partition μ is obtained from λ by removing a sequence of p-hooks. Then we define the *relative sign* $\delta_p(\lambda, \mu)$ by

$$\delta_p(\lambda, \mu)=\delta_p(\lambda)\,\delta_p(\mu).$$

Trivially, the relative sign is transitive. If μ is obtained from λ by removing a sequence of p-hooks, and ζ is obtained from μ by removing a sequence of p-hooks, then $\delta_p(\lambda, \zeta)=\delta_p(\lambda, \mu)\,\delta_p(\mu, \zeta)$. Therefore we are led to consider the minimal case, where μ is obtained from λ by removing a single p-hook H, say. Then a β-set Y for μ is obtained from a β-set for λ by replacing a $c\in X$ by $c-p$ [3, Sect. 2.7]. The leg length of the hook H is then

$$l(H)=|\{d\in X \mid c-p<d<c\}|.$$

Since the number of beads above the one representing c on the relevant runner on the p-abacus for X equals the same number for $c-p$ on the p-abacus for Y, the p-number of $c\in X$ equals the p-number of $c-p\in Y$. But obviously the natural numbering is changed by a cycle of length $l(H)+1$. Thus

LEMMA (2.1). *If μ is obtained by removing a single p-hook H from λ, then*

$$\delta_p(\lambda, \mu) = (-1)^{l(H)},$$

where $l(H)$ is the leg length of the hook.

From this, and the transitivity of the relative sign, we get:

PROPOSITION (2.2). *If μ is obtained from λ by removing a sequence of v p-hooks having leg length $l_1, l_2, ..., l_v$, then*

$$\delta_p(\lambda, \mu) = (-1)^l,$$

where $l = \sum_{i=1}^{v} l_i$. In particular, the residue of l (mod 2) *does not depend on the choice of p-hooks being removed in going from λ to μ.*

This result can be applied in particular to the case where $\mu = \lambda_{(p)}$, the p-core of λ. For a β-set for $\lambda_{(p)}$ the natural numbering and the p-numbering coincide since the beads on the p-abacus are all in the highest possible positions. Thus $\delta_p(\lambda_{(p)}) = 1$ and we have

COROLLARY (2.3). *For any partition*

$$\delta_p(\lambda) = \delta_p(\lambda, \lambda_{(p)}).$$

Note. This shows that $\delta_p(\lambda)$ is the sign σ of Robinson [12] or [3, p. 82].

We shall prove analogues of the above results in the case where $\lambda = (a_1, a_2, ..., a_m)$ is a bar partition and where bars replace hooks. We restrict ourselves to the case where *p is odd.* (For remarks about why this restriction is necessary, see [10, Sect. 2].)

A γ-set $\mathfrak{X}$ is a finite subset of $\mathbb{Z}$ with the following property:

If $i \in \mathfrak{X}$ and $i < 0$ then $i+1 \in \mathfrak{X}$.

The γ-set $\mathfrak{X}$ consists of two disjoint subsets

$$\mathfrak{X}^- = \{i \in \mathfrak{X} \mid i \leqslant 0\}$$

$$\mathfrak{X}^+ = \{i \in \mathfrak{X} \mid i > 0\}.$$

If the elements of $\mathfrak{X}^+$ form the parts of the bar partition λ we call $\mathfrak{X}$ a *γ-set for λ* and write $\bar{P}(\mathfrak{X}) = \lambda$. Thus a bar partition λ has infinitely many γ-sets. These are distinguished simply by $|\mathfrak{X}^-|$ (since $|\mathfrak{X}^-|$ of course determines $\mathfrak{X}^- = \{0, -1, ..., -(|\mathfrak{X}^-|-1)\}$).

EXAMPLE. $\mathfrak{X}=\{-2,-1,0,1,4,5\}$ is a γ-set for $\bar{P}(\mathfrak{X})=\lambda=(5,4,1)\succ\!\!\!-10$.

Our idea is to describe the removal of p-bars in terms of γ-sets which have the same cardinality for these partitions. This will make it possible to define the p-bar sign ($\bar{p}$-sign) of λ as the sign of a permutation.

Suppose then that $\bar{P}(\mathfrak{X})=\lambda$, $\bar{P}(\mathfrak{Y})=\mu$, that $|\mathfrak{X}|=|\mathfrak{Y}|$, and that μ is obtained from λ by removing a p-bar. Then

$$|\mathfrak{Y}^-|=|\mathfrak{X}^-|, \qquad \mathfrak{Y}^+=(\mathfrak{X}^+\setminus\{a\})\cup\{a-p\}, \qquad a>p \quad \text{if the bar is of } \textit{type } 1,$$

$$|\mathfrak{Y}^-|=|\mathfrak{X}^-|+1, \qquad \mathfrak{Y}^+=\mathfrak{X}^+\setminus\{p\} \qquad \text{if the bar is of } \textit{type } 2,$$

$$|\mathfrak{Y}^-|=|\mathfrak{X}^-|+2, \qquad \mathfrak{Y}^+=\mathfrak{X}^+\setminus\{a,a'\}, \qquad (a+a'=p) \quad \text{if the bar is of } \textit{type } 3.$$

As was the case for β-sets, we number the elements of a γ-set $\mathfrak{X}$ by $1,2,...,|\mathfrak{X}|$ in two ways:

In the *natural numbering* the elements of $\mathfrak{X}$ are numbered according to their increasing order. For an odd number p we define the *$\bar{p}$-numbering* of $\mathfrak{X}$ as follows:

The elements of $\mathfrak{X}^+$ are placed on the p-abacus as usual. Two elements of $\mathfrak{X}^+$ are called *conjugate* if

(i) their beads are situated on two conjugate runners,

(ii) they are in the same layer, i.e., the number of beads above these beads on their respective runners is the same.

An element in $\mathfrak{X}^+$ having no conjugate element will be called *isolated.* We then decompose $\mathfrak{X}^+$ as

$$\mathfrak{X}_0^+=\{a\in\mathfrak{X}^+\mid p \text{ divides } a\}$$

$$\mathfrak{X}_c^+=\{a\in\mathfrak{X}^+\mid a \text{ has a conjugate element in } \mathfrak{X}^+\}$$

$$\mathfrak{X}_i^+=\{a\in\mathfrak{X}^+\mid p\nmid a \text{ and } a \text{ is isolated}\}.$$

We first number the elements of $\mathfrak{X}^-$, then those in $\mathfrak{X}_0^+$, then $\mathfrak{X}_c^+$, and finally $\mathfrak{X}_i^+$. The elements of $\mathfrak{X}^-$ are given the same numbers as in the natural numbering. Then the elements of $\mathfrak{X}_0^+$ are numbered in their increasing order. The elements of $\mathfrak{X}_c^+$ are numbered as follows.

(i) Suppose that a and a' are conjugate elements. Then they will be given two consecutive numbers k and $k+1$ with the condition that a will get the lowest number k if and only if the corresponding bead is on a runner with an even number.

(ii) If (a, a') and (b, b') are pairs of conjugate elements, then a and a' are numbered before b and b', if and only if $\min\{a, a'\} < \min\{b, b'\}$.

Finally, the elements of $\mathfrak{X}_{\mathrm{i}}^{+}$ are numbered according to layers as follows: An element represented by the jth *isolated* bead on the ith runner is numbered before the element represented by the j_1th isolated bead on the i_1th runner if and only if $j < j_1$ or $j = j_1$ and $i < i_1$.

When we again compare the natural numbering of $\mathfrak{X}$ with the $\bar{p}$-numbering of $\mathfrak{X}$ we get a permutation $\pi_{\bar{p}}(\mathfrak{X})$, whose sign is denoted by $\delta_{\bar{p}}(\mathfrak{X})$. Since the numbering of the negative elements of $\mathfrak{X}$ is the same in the natural and in the $\bar{p}$-numbering we have that $\delta_{\bar{p}}(\mathfrak{X}) = \delta_{\bar{p}}(\mathfrak{Y})$ if $\mathfrak{X}$ and $\mathfrak{Y}$ are γ-sets with $\mathfrak{X}^{+} = \mathfrak{Y}^{+}$. Therefore we may define

$$\delta_{\bar{p}}(\lambda) := \delta_{\bar{p}}(\mathfrak{X})$$

if $\mathfrak{X}$ is a γ-set with $\bar{P}(\mathfrak{X}) = \lambda$.

Since the above rules and definitions are quite complicated, we illustrate them with an example.

EXAMPLE. $\mathfrak{X} = \{-1, 0, 1, 6, 8, 9, 10, 13, 14, 15, 16, 17, 21\}$. $\bar{P}(\mathfrak{X}) = \lambda = (21, 17, 16, 15, 14, 13, 10, 9, 8, 6, 1)$. Let $p = 5$.

$\mathfrak{X}^{-} = \{-1, 0\}$

$\mathfrak{X}_{0}^{+} = \{10, 15\}$

$\mathfrak{X}_{\mathrm{c}}^{+} = \{1, 6, 8, 9, 14, 17\}$

Conjugate pairs:

$(1, 9), (6, 14), (8, 17)$

$\mathfrak{X}_{\mathrm{i}}^{+} = \{13, 16, 21\}$

5-abacus for $\mathfrak{X}^{+}$

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24

The elements of $\mathfrak{X}^{-}$ are numbered 1, 2 in both numberings. After that we have

Natural numbering

0	1_{3}	2	3	4
5	6_{4}	7	8_{5}	9_{6}
10_{7}	11	12	13_{8}	14_{9}
15_{10}	16_{11}	17_{12}	18	19
20	21_{13}	22	23	24

$\bar{5}$-numbering

0	1_{6}	2	3	4
5	6_{8}	7	8_{10}	9_{5}
10_{3}	11	12	13_{12}	14_{7}
15_{4}	16_{11}	17_{9}	18	19
20	21_{13}	22	23	24

(In the $\bar{5}$-numbering 9 is numbered before 1 because 9 is on the 4th (even) runner and 17 is before 8, since 17 is on the 2nd runner. The layers of $\mathfrak{X}_i^+$ are $\{13, 16\}$, $\{21\}$. Then 16 is before 13, because 16 is on the first runner and 13 on the third.) We have

$$\begin{aligned}\pi_5(\mathfrak{X}) &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 1 & 2 & 6 & 8 & 10 & 5 & 3 & 12 & 7 & 4 & 11 & 9 & 13 \end{pmatrix} \\ &= (3, 6, 5, 10, 4, 8, 12, 9, 7)\end{aligned}$$

and so,

$$\delta_5(\mathfrak{X}) = \delta_5(\lambda) = (-1)^8 = 1.$$

If μ is obtained from λ by removing a sequence of p-bars we define the *relative sign* $\delta_{\bar{p}}(\lambda, \mu)$ as

$$\delta_{\bar{p}}(\lambda, \mu) = \delta_{\bar{p}}(\lambda)\, \delta_{\bar{p}}(\mu).$$

Again we trivially have transitivity. We prove the following:

LEMMA (2.4). *If μ is obtained from λ by removing a single p-bar $\bar{H}$, then*

$$\delta_{\bar{p}}(\lambda, \mu) = (-1)^{l(\bar{H})},$$

where $l(\bar{H})$ is the leg length of the p-bar $\bar{H}$.

Proof. We choose γ-sets $\mathfrak{X}$ and $\mathfrak{Y}$ for λ and μ of the same cardinality and compare the numberings of $\mathfrak{X}$ and $\mathfrak{Y}$. Obviously $\delta_{\bar{p}}(\lambda, \mu)$ will then be the product of the signs of the changes taking place in the natural numbering and in the $\bar{p}$-numbering. We consider separately p-bars of types 1, 2, and 3.

Type 1. Suppose that $a \in \mathfrak{X}^+$ is replaced by $a - p \in \mathfrak{Y}^+$. As in the classical case, the natural numbering is changed by a cycle of length $l(\bar{H}) + 1$. Since a is represented by a bead on the same runner (in the p-abacus for $\mathfrak{X}^+$) as $a - p$ (in the p-abacus for $\mathfrak{Y}$), it is clear that the $\bar{p}$-numbering of $a \in \mathfrak{X}$ is the same as the $\bar{p}$-numbering of $a - p \in \mathfrak{Y}$ unless $a \in \mathfrak{X}_c^+$. In that case the relative position of a and its conjugate a' among the conjugate pairs in $\mathfrak{X}^+$ may be different from that of $a - p$ and a' among the conjugate pairs in $\mathfrak{Y}$. By condition (ii) for the $\bar{p}$-numbering of $\mathfrak{X}_c^+$ ($\mathfrak{Y}_c^+$) we see that the $\bar{p}$-numbering is changed by a product of two cycles of the same length. Thus, (2.4) is true in this case.

Type 2. Suppose that $a = p \in \mathfrak{X}$ is replaced by $0 \in \mathfrak{Y}$. Again the natural numbering is changed by a cycle of length $l(\bar{H}) + 1$, whereas the $\bar{p}$-number of $a \in \mathfrak{X}$ is the same as the $\bar{p}$-number of $0 \in \mathfrak{Y}$ (namely $|\mathfrak{X}^-| + 1 = |\mathfrak{Y}^-|$). So the $\bar{p}$-numbering produces no change.

Type 3. Suppose that $a \in \mathfrak{X}$ is replaced by $-1 \in \mathfrak{Y}$ and that $a' \in \mathfrak{X}$ is replaced by $0 \in \mathfrak{Y}$, where $a + a' = p$, $a < a'$. Let

$$b_1 = |\{a'' \in \mathfrak{X}^+ \mid a'' < a\}|, \qquad b_2 = |\{a'' \in \mathfrak{X}^+ \mid a'' < a'\}|.$$

If we first move a to -1 and then a' to 0 the change in the natural numbering is a product of a cycle of length $b_1 + 1$ with a cycle of length b_2 (not $b_2 + 1$, since $a < a'$ has already been moved). Now (a, a') is a conjugate pair in $\mathfrak{X}^+$. Moreover, a is before a' in the $\bar{p}$-numbering of $\mathfrak{X}$ if and only if a is even. Comparing the $\bar{p}$-numbering of $\mathfrak{X}$ and $\mathfrak{Y}$ we have a change which is a product of two cycles of the same length, if a is even, and which is a product of two cycles, whose lengths differ by 1, if a is odd. Since $b = a + b_2 - (b_1 + 1)$, as is easily seen, we again obtain our result in this case.

Arguing as in the classical case we have the following:

PROPOSITION (2.5). *If μ is obtained from λ by removing a sequence of v p-bars with leg lengths $l_1, l_2, ..., l_v$ then*

$$\delta_{\bar{p}}(\lambda, \mu) = (-1)^{\sum_i l_i}.$$

In particular, the residue of $\sum l_i$ (mod 2) does not depend on the choice of p-bars being removed in going from λ to μ.

If λ is a $\bar{p}$-core and $\mathfrak{X}$ is a γ-set for λ, then $\mathfrak{X}^+ = \mathfrak{X}_{\mathrm{i}}^+$ and so the natural numbering of $\mathfrak{X}$ coincides with the $\bar{p}$-numbering. Thus $\delta_{\bar{p}}(\lambda) = 1$.

COROLLARY (2.6). *If λ is a bar partition, then*

$$\delta_{\bar{p}}(\lambda) = \delta_{\bar{p}}(\lambda, \lambda_{(\bar{p})}).$$

The relative sign is used in Section 4 to describe the sign which occurs by repeated application of the Murnaghan–Nakayama formula for spin characters in which there also occurs a power of 2. By repeated use of (1.3) this gives the "branching coefficient" which is now described.

If λ is a bar partition, $m(\lambda)$, $\varepsilon(\lambda)$, $\lambda^{(\bar{p})} = (\lambda_0, \lambda_1, ..., \lambda_t)$, $t = (p-1)/2$, and $w_{\bar{p}}(\lambda)$ are defined as in Section 1.

Then the *$(\bar{p})$-branching coefficient* of λ is

$$\beta_{\bar{p}}(\lambda) := \left[\frac{w_{\bar{p}}(\lambda) - m(\lambda_0) + \varepsilon(\lambda_{(\bar{p})})}{2}\right],$$

where λ_0 is the first partition in $\lambda^{(\bar{p})}$. (Thus, $m(\lambda_0)$ is the number of parts of λ which are divisible by p.) Obviously the branching coefficient is non-negative.

PROPOSITION (2.7). *Suppose that $\lambda = \mu_0, \mu_1, ..., \mu_w = \lambda_{(\bar{p})}$ is a sequence of bar partitions, such that for each i, $1 \leqslant i \leqslant w = w_{\bar{p}}(\lambda)$, μ_i is obtained by removing a p-bar from μ_{i-1}. Then*

$$\beta_{\bar{p}}(\lambda) = |\{i \mid 0 \leqslant i \leqslant w-1 : \varepsilon(\mu_{i+1}) - \varepsilon(\mu_i) = 1\}|. \qquad (*)$$

Note. $\varepsilon(\mu_{i+1}) - \varepsilon(\mu_i) = 1$ means that μ_{i+1} is odd and μ_i is even. Then if $\mu_{i+1} = \mu_i \backslash \bar{H}_i$, then $m(\bar{H}_i) = 1$ in the notation of Section 1.

Proof. Denote the cardinality of the right-hand side of $(*)$ by β. We obviously only get a contribution to β, when a parity change occurs going from a μ_i to μ_{i+1}, that is, μ_i and μ_{i+1} have different parities. Now, as was seen in Section 1, if the bar being removed going from μ_i to μ_{i+1} is of type 1 or 3, there *is* a parity change, since then

$$(|\mu_i| - m(\mu_i)) - (|\mu_{i+1}| - m(\mu_{i+1})) = \begin{cases} p & \text{by a type 1 bar} \\ p-2 & \text{by a type 3 bar.} \end{cases}$$

There is *no* parity change when a bar of type 2 is being removed. From the description of $\lambda^{(\bar{p})}$ in Section 1 or [10, Sect. 2], it is clear that the total number of bars of type 2 being removed going from λ to $\lambda_{(\bar{p})}$ equals the number of parts in λ_0, that is, $m(\lambda_0)$. Thus the number of parity changes which occur in going from λ to $\lambda_{(\bar{p})}$ is $w^* := w - m(\lambda_0)$, that is, it is independent of the choice of the μ_i. If $\varepsilon(\lambda_{(\bar{p})}) = 1$ we see that the number of changes from even to odd (giving a contribution 1 to β) is $w^*/2$ if w^* is even and $(w^*+1)/2$ if w^* is odd. If $\varepsilon(\lambda_{(\bar{p})}) = 0$, this number is $w^*/2$ if w^* is even and $(w^*-1)/2$ if w^* is odd. Thus the result follows.

If μ is obtained from λ by removing a sequence of p-bars, we define the relative ($\bar{p}$-) branching coefficient as

$$\beta_{\bar{p}}(\lambda, \mu) = \beta_{\bar{p}}(\lambda) - \beta_{\bar{p}}(\mu).$$

Trivially, the relative branching coefficients satisfy a transitivity condition. If we apply (2.7) to λ and μ we have the following:

COROLLARY (2.8). *If μ is obtained from λ by removing a sequence of p-bars, then $\beta_{\bar{p}}(\lambda, \mu)$ is the number of times there is a change from even to odd parity when removing one of these p-bars.*

In particular,

COROLLARY (2.9). *If μ is obtained from λ by removing a single p-bar, then*

$$\beta_{\bar{p}}(\lambda, \mu) = \mathrm{Max}(0, \varepsilon(\mu) - \varepsilon(\lambda)).$$

The results of this section will be applied in an essential way in Section 4.

3. A RESULT ON LEG LENGTHS

Using the p-abacus we give in this section a proof of a quite remarkable result of Robinson on leg lengths of hooks (stated without proof as 4.57 in [12]) and of a similar (but somewhat weaker) result for bars.

Let p be a positive integer and let X be a β-set for the partition λ such that $p \mid |X|$. We may represent the elements of X as beads on the p-abacus. Then the positions of beads on the ith runner represent a β-set for a partition λ_i, $i = 0, 1, 2, ..., p-1$. (See [3, Sect. 2.7] for details.) Then $\lambda^{(p)} = (\lambda_0, \lambda_1, ..., \lambda_{p-1})$ is the *p-quotient* of λ. The removal of an lp-hook in λ is represented on the abacus by moving beads l positions up on a runner. If these moves are on the ith runner this signifies then the removal of an l-hook in λ_i. Therefore if we define a hook in $\lambda^{(p)}$ as a hook in one of the λ_i we have

PROPOSITION (3.1). *Let λ be a partition. There exists a canonical bijection f between the set of hooks of length divisible by p in λ and the set of hooks in $\lambda^{(p)}$. Thereby an lp-hook H of λ is mapped onto an l-hook $f(H)$ in $\lambda^{(p)}$. For the removal of the hooks we then have*

$$(\lambda \backslash H)^{(p)} = \lambda^{(p)} \backslash f(H).$$

The next result is due to Nakayama.

LEMMA (3.2). *Let H be an p-hook in λ. There exists a sequence of partitions*

$$\mu_0 = \lambda, \mu_1, \mu_2, ..., \mu_l = \lambda \backslash H$$

and p-hooks $H_1, H_2, ..., H_l$ such that $\lambda_{j-1} \backslash H_j = \lambda_j$ for $j = 1, 2, ..., l$.

Proof. Suppose that $f(H)$ is a hook in λ_i; so the removal of H is represented by moving a bead A_u on the ith runner l positions up on this runner, say from the kth to the $(k-l)$th row. Suppose that there are beads $A_1, A_2, ..., A_u$ on the ith runner in the rows with the numbers

$$k - l + i_1 < k - l + i_2 < \cdots < k - l + i_u = k,$$

where $1 \leqslant i_1 < i_2 < \cdots < i_u = l$. Then we may slide the bead A_1 to the $(k-l)$th row by i_1 moves of length 1 and then for $j = 2, 3, \ldots, u$ the bead A_j by $i_j - i_{j-1}$ moves of length 1 to the position previously occupied by A_{j-1}. Then the beads $A_1, \ldots, A_u$ are moved a total of l times and by (3.1) each such move corresponds to the removal of a p-hook (via f^{-1}).

Note. It is clear that the decomposition of an lp-hook into l p-hooks is not unique.

EXAMPLE. $\lambda = (6, 5, 2^2)$, $p = 3$, $X = \{11, 9, 5, 4, 1, 0\}$.

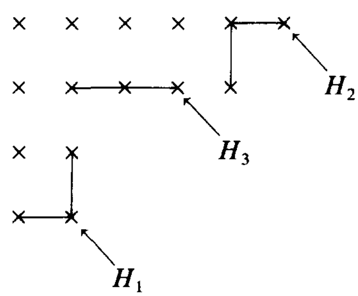
Hook diagram of λ:

9	8	5	4	3	1
7	6	3	2	1	
3	2				
2	1				

The 3-abacus for X:

⓪	①	2
3	④	⑤
6	7	8
⑨	10	⑪

The removal of the 9-hook $(9 = 3 \cdot 3)$ corresponds to the move $11 \to 2$ on the second runner. This move decomposes as $5 \to_{H_1} 2$, $11 \to_{H_2} 8$, $8 \to_{H_3} 5$. In the Young diagram for λ this means



(Another decomposition is $11 \to 8$, $5 \to 2$, $8 \to 5$.)

If $X \subseteq \mathbb{N}_0$, $a, b, i \in \mathbb{N}_0$, we define

$$X(a, b) = \{c \in X \mid a < c < b\}$$

and

$$X(a, b)_p^{(i)} = \{c \in X(a, b),\ c \equiv i \ (\mathrm{mod}\ p)\}.$$

If H is a hook of λ we let $l(H)$ denote its leg length. We remind the reader that $l(H)$ has the following interpretation in terms of a β-set X for λ: suppose that the removal of H is registered in X by replacing $c \in X$ by $c-l$, where l is the length of H; then

$$l(H) = |X(c-l, l)|.$$

(See, e.g., [10, Sect. 1.])

PROPOSITION (3.3) (Robinson). *Let H be a p-hook in λ. Decompose H into l p-hooks $H_1, H_2, ..., H_l$ according to* (3.2). *Then*

$$l(H) - l(f(H)) = \sum_{j=1}^{l} l(H_j).$$

Proof. We use the notation of the proof of (3.2) so that the removal of H is registered by replacing $kp+i \in X$ by $(k-l)\,p+i$. Then the hook of $f(H)$ has leg length $|X((k-l)\,p+i, kp+i)_p^{(i)}|$ by definition. Moreover, the description of the hooks H_j shows that $\sum_j l(H_j)$ equals $|\{c \in X(k-l)\,p+i, kp+i) \mid c \not\equiv i \pmod p\}|$. The result follows.

EXAMPLE. Consider the above example of a 9-hook in $\lambda = (6, 5, 2^2)$. Then $l(H) = 3$, $l(f(H)) = 1$, $l(H_1) = l(H_2) = 1$, $l(H_3) = 0$.

By combining (2.2) and (3.3) we obtain:

COROLLARY (3.4). *Let H be an lp-hook in λ, $\lambda' = \lambda \backslash H$. Then*

$$(-1)^{l(H)} = (-1)^{l(f(H))}\,\delta_p(\lambda, \lambda').$$

We now prove analogues of (3.2), (3.3), and (3.4) for lp-bars in a bar partition, when *p is odd.* In some cases l has to be odd as well. The bar-analogue of (3.1) was proved in [10, Sect. 2]. We refer to that paper for the notation and terminology used in the remainder of this section.

PROPOSITION (3.5). *Let λ be a bar partition. There exists a canonical bijection between the bars of length divisible by p in λ and the set of bars in $\lambda^{(\bar{p})}$ (a bar in $\lambda^{(\bar{p})} = (\lambda_0, \lambda_1, ..., \lambda_t)$ is a bar in λ_0 or a hook in some λ_i, $i = 1, 2, ..., t$). Thereby an lp-bar $\bar{H}$ is mapped onto an l-bar $g(\bar{H})$. Moreover, for the removal of the corresponding bars we have*

$$(\lambda \backslash \bar{H})^{(\bar{p})} = \lambda^{(\bar{p})} \backslash g(\bar{H}).$$

LEMMA (3.6). *Let $\bar{H}$ be an lp-bar in the bar partition λ. There exists a sequence of bar partitions*

$$\mu_0 = \lambda, \mu_1, \mu_2, ..., \mu_l = \lambda \backslash \bar{H}$$

and p-bars $\bar{H}_1, \bar{H}_2, ..., \bar{H}_l$ such that $\mu_{j-1} \backslash \bar{H}_j = \mu_j$ for $j = 1, 2, ..., l$.

Proof. We use the description of the types of bars given in Section 1. For bars of type 1 and 2 we may argue as in (3.2) using (3.5) instead of (3.1). The case of bars of type 3(i) (see [10, Sect. 2]) may also be dealt with in this way using the argument twice (for an $l_1 p$- and $l_2 p$-bar). Consider the case 3(ii) (see [10, Sect. 2]). Suppose that the removal of the bar is indicated by removing the beads (l_1, i) and $(l_2, p-i)$ on the abacus $(l_1 + l_2 + 1 = l)$. (Here, (j, i) means the jth row and the ith column.) Suppose that there are beads in the positions

$$\begin{gathered} (i_{11}, i), (i_{12}, i), ..., (i_{iu}, i) = (l_1, i) \\ (i_{21}, p-i), (i_{22}, p-i), ..., (i_{2v}, p-i) = (l_2, p-i), \end{gathered} \tag{$*$}$$

where $i_{11} < i_{12} < \cdots < i_{1u}$, $i_{21} < i_{22} < \cdots < i_{2v}$. First we move the beads in the positions (i_{11}, i) and $(i_{21}, p-i)$ to the positions $(0, i)$ and $(0, p-i)$ in moves of length 1 (on the given runner). Each such move represents the removal of a p-bar. We have then a partition having i and $(p-i)$ as parts, and so we then remove the mixed p-bar consisting of these parts. Thus we have removed the beads in the positions (i_{11}, i), $(i_{21}, p-i)$ of the configuration $(*)$. Then for $j = 2, 3, ..., u$ we may move the bead in the (i_{1j}, i)-position in moves of length 1 to the (i_{1j-1}, i)-position and proceed similarly on the $(p-i)$th runner. Each of these moves corresponds again to the removal of a p-bar by (3.5). We have then removed $l_1 + l_2 + 1 = l$ p-bars from λ to go to $\lambda \backslash \bar{H}$, as desired.

We denote the leg length of a bar $\bar{H}$ by $l(\bar{H})$.

PROPOSITION (3.7). *Let $\bar{H}$ be an lp-bar in λ, where lp is odd. Decompose $\bar{H}$ into l p-bars $\bar{H}_1, ..., \bar{H}_l$ according to* (3.6). *Then*

$$l(\bar{H}) - l(g(\bar{H})) \equiv \sum_{j=1}^{l} l(\bar{H}_j) \qquad (\text{mod } 2).$$

Note. Contrary to (3.3) there are easy examples to show that equality does not hold in (3.7). Since we are only interested in relative signs, the result is sufficient for our purposes. It should be noted that if all the bars $\bar{H}$ are of type 1 and 2 we have equality.

Proof. In the cases of bars of type 1 and 2 we argue as in (3.3) (and in fact obtain equality).

Type 3(i). We consider λ also as the subset of $\mathbb{N}_0$ consisting of the parts of λ and use the notation mentioned before (3.3). Suppose that

$$|\lambda(0, l_1 p)_p^{(0)}| = \alpha, \qquad |\lambda(l_1 p, l_2 p)_p^{(0)}| = \beta$$
$$|\lambda(0, l_1 p)| = \alpha + a, \qquad |\lambda(l_1 p, l_2 p)| = \beta + b,$$

so that $\alpha, \beta, a, b \geqslant 0$. By definition

$$l(\bar{H}) = l_1 p + (\beta + b)$$

so that

$$l(\bar{H}) \equiv l_1 + (\beta + b) \pmod 2 \tag{1}$$

since p is odd. Also by definition $l(g(\bar{H})) = l_1 + \beta$ (since $g(\bar{H})$ is a mixed bar in λ_0) and so by (1)

$$l(\bar{H}) - l(g(\bar{H})) \equiv b \pmod 2. \tag{2}$$

If we decompose the $l_1\, p$-bar into l_1 p-bars as before we obtain for the leg lengths of those bars

$$\sum' l(\bar{H}_j) = a. \tag{3}$$

When we decompose the $l_2\, p$-bar into l_2 p-bars we obtain for the leg lengths of those bars

$$\sum'' l(\bar{H}_j)k = a + b \tag{4}$$

so that by (3) and (4) we have

$$\sum_j l(\bar{H}_j) = 2a + b \equiv b \pmod 2.$$

Type 3(ii). This case is much more delicate and requires a division into two subcases. Suppose first that:

$l_1 > l_2$: The lp-bar $\bar{H}$ consists of the parts $l_1 p + i$ and $l_2 p + (p - i)$ of λ, where $l_1 p + i > l_2 p + (p - i)$. Suppose that

$$|\lambda(0, l_1 p)_p^{(i)}| = \alpha, \qquad |\lambda(0, l_2 p)_p^{(p-i)}| = \beta$$
$$|\lambda((l_2 + 1) p, l_1 p)_p^{(i)}| = \gamma, \qquad |\lambda(l_2 p, l_1 p)_p^{(p-i)}| = \delta$$

and let

$$|\lambda(l_2 p + (p - i), l_1 p)| = a + \gamma + \delta - 1.$$

(Note: $\gamma\ a \geqslant 0$, $\delta \geqslant 1$.) By definition

$$l(\bar{H}) = l_2 p + (p-i) + a + \gamma + \delta - 1$$

and so

$$l(\bar{H}) \equiv l_2 + i + a + \gamma + \delta \qquad (\text{mod } 2). \tag{5}$$

In this case $g(\bar{H})$ is a hook in λ_i and from the description of λ_i in [10, Sect. 2], we get $l(g(\bar{H})) = \alpha + l_2 - \beta$, and so by (5)

$$l(\bar{H}) - l(g(\bar{H})) \equiv i + a + \alpha + \beta + \gamma + \delta \qquad (\text{mod } 2). \tag{6}$$

Consider then the leg lengths of the bars $\bar{H}_1, ..., \bar{H}_l$ and suppose that $\bar{H}_v$ (say) is the unique mixed bar amongst these (see the proof of (3.6)). For $j \neq v$ the removal of $\bar{H}_j$ is visualized on the p-abacus by moving a bead one position up on the same runner and $l(\bar{H}_j)$ equals the number of beads on the abacus overtaken in this move. We decompose

$$l(\bar{H}_j) = l_j^0 + l_j^1,$$

where $l_j^0 = 1$ or 0 depending on whether a bead on the conjugate runner is overtaken or not and $l_j^1 \geqslant 0$. Similarly we decompose

$$l(\bar{H}_v) = l_v^0 + l_v^1,$$

where $l_v^0 = i$ and l_v^1 equals the number of beads between i and $(p-i)$ in μ_{v-1}.

Then $\sum_{j=1}^{l} l_j^1$ counts the beads on the p-abacus of λ outside the ith and the $(p-i)$th runner between $l_2 p + (p-i)$ and $l_1 p + i$ once and the beads outside the ith and the $(p-i)$th runner between 0 and $l_2 p + (p-i)$ twice. Thus

$$\sum_{j=1}^{l} l_j^1 \equiv a \qquad (\text{mod } 2) \tag{7}$$

By (6) and (7) it remains to show that

$$\sum_{j=1}^{l} l_j^0 \equiv i + \alpha + \beta + \gamma + \delta \qquad (\text{mod } 2)$$

or equivalently

$$d \equiv \alpha + \beta + \gamma + \delta \qquad (\text{mod } 2), \tag{8}$$

where

$$d=\sum_{\substack{j=1\\ j\neq v}}^{l} l_j^0.$$

By definition d depends only on the positions of the beads on the ith and $(p-i)$th runners. We related d to a relative sign $\delta_2(\mu, \tilde{\mu})$ for suitable partitions μ and $\tilde{\mu}$ (see Section 2).

We extend the ith and the $(p-i)$th runner by adding an empty position before the 0th row and consider these two extended runners with their beads as a 2-abacus of an (ordinary) partition μ. More explicitly, if

$$A=\{[x/p] \mid x\in\lambda_p^{(i)}\}, \qquad B=\{[x/p] \mid x\in\lambda_p^{(p-i)}\}$$

then

$$Y=\{2a+2 \mid a\in A\}\cup\{2b+3 \mid b\in B\}$$

is a β-set for μ.

If $1<v$ then the removal of $\bar{H}_1$ is registered by moving a bead one position up on the ith or on the $(p-i)$th runner (for λ). Moving the corresponding bead one position up on the 2-abacus for μ corresponds to the removal of a 2-hook in μ (to get μ' say). Proceeding like this we get a sequence of partitions $\mu=\mu^0, \mu^1, ..., \mu^{v-1}$, such that each is obtained from the preceding by removing a 2-hook. We register the removal of the mixed p-bar $\bar{H}_v$ in the β-set for μ^{v-1} by replacing 3 by 1 and 2 by 0. Then the partition μ^v having this β-set is obtained from μ^{v-1} by removing two 2-hooks. With the remaining bars $\bar{H}_{v+1}, ..., \bar{H}_l$ we proceed as with $\bar{H}_1, ..., \bar{H}_{v-1}$ to get a sequence of partitions $\mu^{v+1}, ..., \mu^l$. Then $\mu^l=\tilde{\mu}$ is obtained from μ by removing a total of $(l+1)$ 2-hooks. We claim that $(-1)^d=\delta_2(\mu, \tilde{\mu})$, the relative sign. Indeed, for $j\neq v$, l_j^0 is the leg length of the relevant hook, by definition. Moreover, the leg lengths of the two 2-hooks being removed going from μ^{v-1} to μ^v are the same. Therefore our claim follows from (2.2).

But $\tilde{\mu}$ can also be obtained from μ by removing first a $(2l_1+2)$-hook H_1 (corresponding to replacing $2l_1+2\in Y$ by 0) and then a $(2l_2+2)$-hook H_2 (corresponding to replacing $2l_2+3$ by 1). If $\tilde{\tilde{\mu}}=\mu\backslash H_1$ then by the transitivity of the relative sign

$$\delta_2(\mu, \tilde{\mu})=\delta_2(\mu, \tilde{\tilde{\mu}})\,\delta_2(\tilde{\tilde{\mu}}, \tilde{\mu}).$$

Using (3.4) we have

$$\delta(\mu, \tilde{\tilde{\mu}})=(-1)^{l(H_1)}\,(-1)^{l(f(H_1))}$$

$$\delta(\tilde{\tilde{\mu}}, \tilde{\mu})=(-1)^{l(H_2)}\,(-1)^{l(f(H_2))}$$

or

$$d \equiv l(H_1) + l(f(H_1)) + l(H_2) + l(f(H_2)) \qquad (\text{mod } 2). \tag{9}$$

From the definition of H_1 and H_2 we get easily

$$l(H_1) = \alpha + \beta + \delta, \qquad l(f(H_1)) = \alpha$$
$$l(H_2) = \alpha - \gamma + \beta, \qquad l(f(H_2)) = \beta$$

so that by (9)

$$d \equiv \alpha + \beta + \gamma + \delta \qquad (\text{mod } 2)$$

as desired.

The reader may have noticed that it was not used above that l is odd. However, this assumption is necessary in the next case. Suppose that:

$l_1 \leqslant l_2$: The lp-bar $\bar{H}$ consists of the parts $l_1 p + i$ and $l_2 p + (p - i)$ of λ, where $l_1 p + i < l_2 p + (p - i)$. Suppose that

$$|\lambda(0, l_1 p)_p^{(i)}| = \alpha, \qquad |\lambda(0, l_2 p)_p^{(p+i)}| = \beta$$
$$|\lambda(l_1 p, l_2 p)_p^{(i)}| = \gamma, \qquad |\lambda(l_1 p, l_2 p)_p^{(p-i)}| = \delta$$

and let

$$|\lambda(l_1 p + i, l_2 p + (p - i))| = a + \gamma + \delta - 1,$$

so that $a \geqslant 0$, $\gamma \geqslant 1$, $\delta \geqslant 0$. By a similar argument as above we get

$$l(\bar{H}) - l(g(\bar{H})) \equiv a + (l_1 + l_2) + \alpha + \beta + \gamma + \delta + i + 1 \qquad (\text{mod } 2). \tag{6$'$}$$

Defining l_j^0 and l_j^1 and d in analogy with the above we are reduced to show

$$d \equiv l_1 + l_2 + \alpha + \beta + \gamma + \delta + 1 \qquad (\text{mod } 2). \tag{8$'$}$$

Since $l = l_1 + l_2 + 1$ is odd we get $l_1 + l_2 \equiv 0$ (mod 2). Letting μ, $\tilde{\bar{\mu}}$, $\tilde{\mu}$, H_1, and H_2 be defined similarly to the above we get in this case

$$l(H_1) = \alpha + \beta - \delta, \qquad l(f(H_1)) = \alpha$$
$$l(H_2) = \alpha + \beta + \gamma - 1, \qquad l(f(H_2)) = \beta,$$

The relative sign argument then shows that (8)$'$ holds.

COROLLARY (3.8). *Let $\bar{H}$ be an lp-bar in λ, $\tilde{\lambda} = \lambda \backslash \bar{H}$. Then*

$$(-1)^{l(\bar{H})} = (-1)^{l(g(\bar{H}))}\, \delta_{\bar{p}}(\lambda, \tilde{\lambda}).$$

4. The p-Bar Quotient Character

We define certain induced characters and show that they have a property similar to the p-quotient character in symmetric groups.

First we state a trivial property of an induced character for later reference. Let ψ be a character of the subgroup H of G, $x \in G$. Then

$$\psi^G(x) = \sum_{yH \in [X \backslash H]} \psi(y^{-1}xy), \tag{1}$$

where yH runs through the set of cosets of H contained in $X = \{y \in G \mid y^{-1}xy \in H\}$.

In the rest of this section we consider a *fixed* spin character $\langle \lambda \rangle$ of $\hat{S}_n$, $\lambda \succ\!\!- n$. Let p be an odd number and consider the $\bar{p}$-core $\lambda_{(\bar{p})} = \mu$ and the $\bar{p}$-quotient $\lambda^{(\bar{p})} = (\lambda_0, \lambda_1, ..., \lambda_t)$, $t = (p-1)/2$. Suppose that $\lambda_0 \succ\!\!- w_0$, $\lambda_i \vdash w_i$, $i = 1, 2, ..., t$, and $w = w_{\bar{p}}(\lambda) = w_0 + w_1 + \cdots + w_t$, $w' = w - w_0$. We assume that

$$r = |R|, \qquad R = \{i \mid 1 \leqslant i \leqslant t,\ w_i \text{ odd}\},$$

$$s = |S|, \qquad S = \{i \mid 1 \leqslant i \leqslant t,\ w_i \text{ even}\},$$

and so $r + s = t$.

Let H be the inverse image of the Young subgroups $S_{w_0} \times S_{w_1} \times \cdots \times S_{w_t}$ of S_w in $\hat{S}_w$.

Lemma (4.1). *There is a character ϕ of H so that if $\pi_i \in P^0(w_i)$, $i = 0, ..., t$, then*

$$\phi(\pi_0 \circ \pi_1 \circ \cdots \circ \pi_t) = \langle \lambda_0 \rangle(\pi_0) \prod_{i=1}^{t} \langle \hat{w}_i \rangle(\pi_i).$$

Proof. By [13, Sects. 26–28], there is an (irreducible) character χ of H, such that

$$\chi(\pi_0 \circ \pi_1 \circ \cdots \circ \pi_t) = 2^{[(s+\varepsilon(\lambda))/2]} \langle \lambda_0 \rangle(\bar{\pi}_0) \prod_{i=1}^{t} \langle w_i \rangle(\bar{\pi}_i).$$

But an easy calculation shows that

$$\phi := 2^{s - [(s+\varepsilon(\lambda_0))/2]} \chi = 2^{[(s+1-\varepsilon(\lambda_0))/2]} \chi$$

is a character with the desired property.

Note. The *degree* of ϕ in (4.1) is

$$\phi(1) = 2^{(w'-r)/2} \langle \lambda_0 \rangle(1). \tag{2}$$

Proof. Obviously $\phi(1) = \langle \lambda_0 \rangle(1) \prod_{i=1}^{t} \langle \hat{w}_i \rangle(1)$. Moreover, $\langle w_i \rangle$ (and $\langle w_i \rangle'$) has degree $2^{[(w_i-1)/2]}$, and so by definition

$$\langle \hat{w}_i \rangle(1) = \begin{cases} 2^{(w_i-1)/2} & \text{if } \ w_i \text{ is odd}, \\ 2^{w_i/2} & \text{if } \ w_i \text{ is even}. \end{cases}$$

Since r of the w_i are odd, the result follows.

Now $\phi' = [w_0] \otimes [\lambda_1] \otimes \cdots \otimes [\lambda_t]$ is a character of $S_{w_0} \times S_{w_1} \times \cdots \times S_{w_t}$ and thus a character of H. Let

$$\psi = \phi\phi'$$

so that if $\pi_i \in P^0(w_i)$, $0 \leqslant i \leqslant t$, then (see paragraph preceding (1.6))

$$\psi(\pi_0 \circ \pi_1 \circ \cdots \circ \pi_t) = \langle \lambda_0 \rangle(\pi_0) \prod_{i=1}^{t} \langle \hat{\lambda}_i \rangle(\pi_i). \tag{3}$$

We want to specify how to compute the character value in (3) using Murnaghan–Nakayama-type formulae. Indeed, by applying (1.3) to $\langle \lambda_0 \rangle$ and (1.6) to the $\langle \lambda_i \rangle$, $1 \leqslant i \leqslant t$, we obtain the following: Suppose $\pi_i = (l_{i1}, l_{i2}, ..., l_{iv_i})$. Then

$$\psi(\pi_0 \circ \pi_i \circ \cdots \circ \pi_t) = \sum_{(T_0, T_1, ..., T_t)} 2^{\tilde{\beta}(T_0, ..., T_t)}(-1)^{\sum_i l(T_i)}, \tag{4}$$

where:

(i) T_0 is a sequence of bars of length $l_{01}, ..., l_{0v_0}$ whose removal reduces λ_0 to 0.

(ii) For $1 \leqslant i \leqslant t$, T_i is a sequence of hooks of length $l_{i1}, ..., l_{iv_i}$ whose removal reduces λ_i to 0.

(iii) $\tilde{\beta}(T_0, ..., T_t) = d(T_0) + d(T_1) + \cdots + d(T)$, where $d(T_0)$ is the number of changes from a partition of even parity to a partition of odd parity when removing the bars in T_0 and for $1 \leqslant i \leqslant t$, $d(T_i)$ is the number of changes from a partition of even cardinality to a partition of odd cardinality when removing the hooks in T_i.

(iv) $l(T_i)$ is the sum of the leg lengths of the bars (or hooks) in T_i.

We claim that

$$\tilde{\beta}(T_0, T_1, ..., T_t) = \frac{v - r - \varepsilon(\lambda_0) - m_0^*}{2}, \tag{5}$$

where $v = \sum_{i=0}^{t} v_i$ and m_0^* is the number of bars of type 2 in T_0. Indeed, there are $v_0 - m_0^*$ parity changes in removing the bars in T_0 (see the proof of (2.7) for an argument which shows this). Since the removal of the bars reduces λ_0 to 0, the parity $\varepsilon(\lambda_0) \equiv v_0 - m_0^*$ (mod 2) and we see that

$$d(T_0) = \frac{v_0 - m_0^* - \varepsilon(\lambda_0)}{2}.$$

Let $1 \leqslant i \leqslant t$. Trivially the removal of each hook in T_i gives a parity change in the cardinality of the partitions. Thus

$$d(T_i) = \begin{cases} [(v_i+1)/2] & \text{if} \quad w_i \text{ is even,} \\ [v_i/2] & \text{if} \quad w_i \text{ is odd.} \end{cases}$$

But since all parts of π_i are odd, we have $w_i \equiv v_i$ (mod 2). Thus, we have

$$d(T_i) = \begin{cases} v_i/2 & \text{if} \quad w_i \text{ is even,} \\ (v_i - 1)/2 & \text{if} \quad w_i \text{ is odd.} \end{cases}$$

From this (5) follows.

We now define the *p-bar quotient character* $\langle \lambda^{(\bar{p})} \rangle$ as the induced character

$$\langle \lambda^{(\bar{p})} \rangle = \psi^{\tilde{S}_w}, \tag{6}$$

where ψ is as in (3).

Note: The degree of $\langle \lambda^{(\bar{p})} \rangle$ is

$$\langle \lambda^{(\bar{p})} \rangle (1) = 2^{[(w - m(\lambda_0) - r)/2]} w! / \bar{h}(\lambda_0) \prod_{i=1}^{t} h(\lambda_i), \tag{7}$$

where $\bar{h}(\lambda_0)$ is the product of the bar lengths in λ_0 and $h(\lambda_i)$ is the product of the hook lengths in λ_i, $1 \leqslant i \leqslant t$.

This follows by an easy calculation from (2) and the degree formulae for $\langle \lambda_0 \rangle (1)$ and $[\lambda_i](1)$ (see [8, Theorem 1; 3, 2.3.21]).

We now state our main result.

THEOREM (4.2). *The notation is that introduced above. Let* $k = \varepsilon(\mu) + \varepsilon(\lambda_0) + r$. *Suppose that* $\rho \in P^0(w)$, $\tilde{\rho} = p\rho \in P^0(pw)$, $\pi \in P^0(n - pw)$. *Then* $\langle \lambda \rangle (\tilde{\rho} \circ \pi) = \delta_{\bar{p}}(\lambda)\, 2^{[k/2]} \langle \lambda^{(\bar{p})} \rangle (\rho) \langle \mu \rangle (\pi)$.

As a corollary of this, using $\rho = (1^w)$ we obtain a result which was stated in a less precise form in [2] (without a detailed proof). This result will be applied in [11].

COROLLARY (4.3). *Let $\pi \in P^0(n-pw)$. Then*

$$\langle\lambda\rangle((p^w)\circ\pi) = \delta_{\bar{p}}(\lambda)\, 2^{\beta_{\bar{p}}(\lambda)} w!\langle\mu\rangle(\pi)/\bar{h}(\lambda_0) \prod_{i=1}^{t} h(\lambda_i).$$

Proof of (4.3). Put $\rho = (1^w)$ in (4.2) and use (7) to obtain

$$\begin{aligned}\langle\lambda\rangle((p^w)\circ\pi) &= \delta_{\bar{p}}(\lambda)\, 2^{[h/2]}\langle\lambda^{(\bar{p})}\rangle(1)\langle\mu\rangle(\pi)\\ &= \delta_{\bar{p}}(\lambda)\, 2^{[k/2]}\cdot 2^{[(w-m(\lambda_0)-r)/2]} w!\langle\mu\rangle(\pi)/\bar{h}(\lambda_0)\prod_{i=1}^{t} h(\lambda_i).\end{aligned}$$

By definition (see Section 2)

$$\beta_{\bar{p}}(\lambda) = [(w-m(\lambda_0)+\varepsilon(\mu))/2].$$

But since $w'-r$ is even and $[(w_0-m(\lambda_0))/2] = (w_0-m(\lambda_0)-\varepsilon(\lambda_0))/2$ we obtain

$$\begin{aligned}[k/2]+[(w-m(\lambda_0)-r)/2] &= [k/2]+(w-m(\lambda_0)-\varepsilon(\lambda_0)-r)/2\\ &= [(\varepsilon(\mu)+\varepsilon(\lambda_0)+r+w-m(\lambda_0)-\varepsilon(\lambda_0)-r)/2]\\ &= \beta_{\bar{p}}(\lambda),\end{aligned}$$

as required.

Proof of (4.2). Let $\rho = (l_1, l_2, ..., l_v)$, where $l_1 \geqslant l_2 \geqslant \cdots \geqslant l_v$ and the l_i are odd, hence $p\rho = (pl_1, pl_2, ..., pl_v)$. Now μ is the only partition of $n-pw$ which can be obtained from λ by removing w p-bars. Therefore, using (3.6) we see that μ is also the only partition which can be obtained by removing a sequence $\bar{H}_1, \bar{H}_2, ..., \bar{H}_v$ of bars, where each $\bar{H}_i$ is a pl_i-bar. By using the Murnaghan–Nakayama formula for spin characters (1.3) v times, we have

$$\langle\lambda\rangle(\tilde{\rho}\circ\pi) = \sum_{(\bar{H}_1, \bar{H}_2, ..., \bar{H}_v)} (-1)^{\sum l(\bar{H}_i)} 2^{\beta(\bar{H}_1, ..., \bar{H}_v)}\langle\mu\rangle(\pi), \tag{8}$$

where the sum is over the set H of all v-tuples $(\bar{H}_1, ..., \bar{H}_v)$ of bars satisfying

$$\bar{H}_1 \text{ is a } pl_1\text{-bar in } \mu_1 = \lambda,\ \bar{H}_2 \text{ is a } pl_2\text{-bar in } \mu_2 = \mu_1\backslash\bar{H}_1, \text{ etc.} \tag{$*$}$$

(so that $\mu = \mu_{v+1} = \mu_v\backslash\bar{H}_v$). Moreover $l(\bar{H}_i)$ is the leg length of the bar $\bar{H}_i$ and

$$\begin{aligned}\beta(\bar{H}_1, ..., \bar{H}_v) &= |\{i \mid \mu_i \text{ is even and } \mu_{i+1} \text{ is odd}\}|\\ &= |\{i \mid \varepsilon(\mu_{i+1}) - \varepsilon(\mu_i) = 1\}|.\end{aligned} \tag{9}$$

Arguing as in the proof of (2.7) we see that

$$\beta(\bar{H}_1, ..., \bar{H}_v) = [(v - m^* + \varepsilon(\mu))/2], \tag{10}$$

where m^* is the number of bars of type 2 among the $\bar{H}_i$.

Consider a tuple $T = (\bar{H}_1, ..., \bar{H}_v) \in \mathfrak{H}$ and apply the map g of (3.5) to each $\bar{H}_i$ to get a tuple $g(T) = (g(\bar{H}_1), ..., g(\bar{H}_v))$, where now for all i, $1 \leqslant i \leqslant v$,

> $g(\bar{H}_i)$ is either an l_i-bar in the first partition of $\mu_i^{(\bar{\rho})}$ or an l_i-hook in one of the other partitions of $\mu_i^{(\bar{\rho})}$ and $\mu_{i+1}^{(\bar{\rho})} = \mu_i^{(\bar{\rho})} \setminus g(\bar{H}_i)$. $(**)$

Thus T induces a map θ_T: $\{1, 2, ..., v\} \to \{0, 1, ..., t\}$ by $\theta_T(i) = j$ if $g(\bar{H}_i)$ is in the jth partition of $\mu_i^{(\bar{\rho})}$.

If $x = x_1 x_2 \cdots x_v$ is an element of odd order of type ρ in $\hat{S}_w$ such that x_i corresponds to a cycle of length l_i we get also from T a coset $y_T - H \in [X \setminus H]$, where $X = \{y \in \hat{S}_w \mid y^{-1}xy \in H\}$ (see the beginning of this section). We choose y_T as an element which conjugates the x_i disjointly into the $\theta_T(i)$th factor in H. Since we are only interested in $y_T H$, it does not matter how this is done. Now let

$$T_j = (g(H_i) \mid \theta_T(i) = j), \qquad 0 \leqslant j \leqslant t$$

(where the order is the natural one coming from $\{1, 2, ..., v\}$). We see that $(T_0, T_1, ..., T_t)$ satisfies (i) and (ii) in the conditions following (4). So from $T = (\bar{H}_1, ..., \bar{H}_v) \in \mathfrak{H}$ we get the data $(y_T H; T_0, T_1, ..., T_t)$. However, from this data we may recover T. (If $y_T^{-1} x_1 y_T \in \hat{S}_{w_{\theta(1)}}$ let $\bar{H}_1$ be the bar corresponding under g to the first bar (hook) in $T_{\theta(1)}$. If $y_T^{-1} x_2 y_T \in \hat{S}_{w_{\theta(2)}}$ and $\theta(2) \neq \theta(1)$, then $\bar{H}_2$ is the bar corresponding to the first bar or hook in $T_{\theta(2)}$. If $\theta(2) = \theta(1)$, $\bar{H}_2$ corresponds to the second bar or hook in $T_{\theta(1)}$, etc.)

Thus each $T \in \mathfrak{H}$ corresponds uniquely to an element $yH \in [X \setminus H]$ with $(T_0, T_1, ..., T_t)$. Now $(T_0, T_1, ..., T_t)$ is a summation index, if we compute $\psi(y_T^{-1} x y_T)$ using (4). (In (4), $\rho_j = (l_i^1 \mid \theta_T(i) = j)$.) This means that we have given a bijection between the summands in the value $\langle \lambda \rangle (\tilde{\rho} \circ \pi)$ computed using (8) and the summands in the value $\langle \lambda^{(\bar{\rho})} \rangle (\rho)$ computed using (1) and (4).

If $(\bar{H}_1, ..., \bar{H}_v) \leftrightarrow (yH; T_0, ..., T_t)$ we have to compare

$$\beta(H_1, ..., H_v) \quad \text{and} \quad \tilde{\beta}(T_0, ..., T_t)$$

and

$$(-1)^{\sum l(\bar{H}_i)} \quad \text{and} \quad (-1)^{\sum l(T_i)}.$$

The latter is easily done using (3.8) and the transitivity of the relative sign. In any case we obtain

$$(-1)^{\sum l(\bar{H}_i)} = \delta_{\bar{p}}(\lambda)(-1)^{\sum l(T_i)}. \tag{11}$$

To compare $\beta(\bar{H}_1, ..., \bar{H}_v)$ and $\tilde{\beta}(T_0, ..., T_t)$ we use (10) and (5). By the description of g given in [10, (2.3) and the proof] we have that $m_0 = m_0^*$. Subtracting Eq. (10) from Eq. (5) we then obtain

$$\beta(\bar{H}_1, ..., \bar{H}_v) - \tilde{\beta}(T_0, T_1, ..., T_t) = [k/2]. \tag{12}$$

Now (11) and (12) show that the "differences" between $(-1)^{\sum l(\bar{H}_i)}$ and $(-1)^{\sum l(T_i)}$ and between $\beta(\bar{H}_1, ..., \bar{H}_v)$ and $\tilde{\beta}(T_0, ..., T_t)$ are *independent* of the bars $\bar{H}_1, ..., \bar{H}_v$. Therefore our result is proved.

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